Solution Set 5, 18.06 Fall '12

1. Do problem 5 from 4.1

   Solution. (a) $Ax = b$ is saying that b is in the column space of A, $A^T y = 0$ is saying that y is in the nullspace of A^T, we know that the column space of A is orthogonal to the null space of A^T, so we have $y^T b = 0$ (note that y and x don't necessarily have the same dimension so the second option doesn't really make sense).

   (b) From the information given, we know that $(1, 1, 1)^T$ is in the column space of A^T and x is in the null space of A. This implies that x is orthogonal to $(1, 1, 1)$. $\square$

2. Do problem 29 from 4.1

   Solution. If we write $AA^{-1} = I$ in terms of rows of A and columns of A^{-1}, we see that the first column of A^{-1} is orthogonal to all the rows of A except the first. Indeed the inner product of the first column of A^{-1} with the first row of A is 1. $\square$

3. Do problem 29 from 4.1

   Solution. For the first question we can take :

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 0 & 0 \\ 3 & 0 & 0 \end{pmatrix}$$

   Clearly, v is in the row space and column space.

   For the second quesion we can take

$$A = \begin{pmatrix} -2 & 1 & 0 \\ -4 & 2 & 0 \\ -6 & 3 & 0 \end{pmatrix}$$

   It is easy to check that v is in the column space and in the null space.

   Clearly v is not orthogonal to itself (the only vector that is orthogonal to itself is 0) therefore v can't be both in the null space of A and the row space or both in the null space of A^T and in the column space. $\square$

4. Do problem 13 from 4.2

   Solution. The column space of A is easily described as the space of 4-dimensional vectors whose last coordinate is 0. The projection of b is the vector $p(b) = (1,2,3,0)$, indeed $p(b)$ must be in the column space of A and be such that $b - p(b)$ is orthogonal to the column space of A. Clearly $(1,2,3,0)$ satisfies these two conditions.

   Using the same reasoning, it is not hard to check that projecting any vector to the column space of A is the same thing as setting the last coordinate equal to 0.

   If we believe that, the projection matrix is $P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$. Indeed when this P is multiplied with a vector, it has exacly the effect of leaving the first three coordinates the same and setting the last one to 0.

   One could also use the formula $P = A(A^TA)^{-1}A^T$.

   If we use this, we find that A^TA is the 3 by 3 identity matrix. So $P = AI^{-1}A^T = AA^T$ and then it is a straightforward computation to check that P is what is written above. □

5. Do problem 22 from 4.2

   Solution. We just compute. The key points to have in mind is that $(M^{-1})^T = (M^T)^{-1}$, $(MN)^T = N^TM^T$ and $(M^T)^T = M$:

$$\begin{aligned} P^T &= (A(A^TA)^{-1}A^T)^T = A(((A^TA)^{-1})^T)A^T \\ &= A(((A^TA)^T)^{-1})A^T = A(A^TA)^{-1})A^T = P \end{aligned}$$

□

6. Do problem 24 from 4.2

   Solution. The nullspace of A^T is orthogonal to the column space of A. So if $A^Tb = 0$, the projection of b onto $C(A)$ should be $p = 0$. Indeed the projection of a vector onto a space it is orthogonal to is 0.

   If we take $P = A(A^TA)^{-1}A^T$, it is obvious that we indeed have $Pb = 0$. □

7. Do problem 30 from 4.2

Solution. (a) We see that the column space of A is the line generated by $a = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$. The projection matrix is then $P_C = aa^T/a^Ta$. We find :

$$P_C = \frac{1}{25}\begin{pmatrix} 9 & 12 \\ 12 & 16 \end{pmatrix}$$

(b) The row space of A is the line generated by $\begin{pmatrix} 3 \\ 6 \\ 6 \end{pmatrix}$ which is the same as the line generated by $\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$. We use the same formula that we used in the first question :

$$P_R = \frac{1}{9}\begin{pmatrix} 1 & 2 & 2 \\ 2 & 4 & 4 \\ 2 & 4 & 4 \end{pmatrix}$$

If we do the computation $P_C A P_R$ we find A. Let's try to explain that.

First let's try to understand what $P_C A$ is. The column of the matrix $P_C A$ are P_C times the columns of A. But P_C is the projection onto the column space and the columns of A are in the column space by definition. Hence we see that P_C does nothing to the columns of A and $P_C A = A$.

Now we are reduced to understand AP_R. It is easier to try to understand its transpose $P_R^T A^T$. We know that $P_R^T = P_R$, so we need to understand $P_R A^T$. The columns of this matrix are P_R times the columns of A^T but the columns of A^T are the rows of A. Hence they are in the row space and multiplying them by P_R leaves them unchanged. This is precisely saying that $P_R A^T = A^T$. However we were really interested in AP_R which is the transpose of what we have just computed. Thus $AP_R = A$.

This explains why $P_C A P_R = A$. □

8. Do problem 5 from 4.3

Solution. A horiszontal line of height C fitting these 4 points would satisfy :

$$\begin{aligned} C &= 0 \\ C &= 8 \\ C &= 8 \\ C &= 20 \end{aligned}$$

In matrix form, we want to solve :

$AC = b$ with $A = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$. Of course there are no exact solutions but the best approximation is given by solving the equation:

$$A^T AC = A^T b$$

This equation becomes $4C = 0 + 8 + 8 + 20 = 36$. The best C is 9. I didn't draw the picture but the errors are the difference between the actual value and 9 i.e. $9, 1, 1, 11$. □

9. Do problem 17 from 4.3

Solution. The equations we need to solve if there was an exact solution is :

$$\begin{aligned} C - D &= 7 \\ C + D &= 7 \\ C + 2D &= 21 \end{aligned}$$

Of course this system is unsolvable. We know that the best approximation to a solution is given by the solution to $A^T A \begin{pmatrix} C \\ D \end{pmatrix} = A^T \begin{pmatrix} 7 \\ 7 \\ 21 \end{pmatrix}$ with $A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \\ 1 & 2 \end{pmatrix}$.

We have $A^T A = \begin{pmatrix} 3 & 2 \\ 2 & 6 \end{pmatrix}$ and $A^T \begin{pmatrix} 7 \\ 7 \\ 21 \end{pmatrix} = \begin{pmatrix} 35 \\ 42 \end{pmatrix}$. Therefore we want to solve :

$$\begin{pmatrix} 3 & 2 \\ 2 & 6 \end{pmatrix} \begin{pmatrix} C \\ D \end{pmatrix} = \begin{pmatrix} 35 \\ 42 \end{pmatrix}$$

We find $C = 9$ and $D = 4$. Hence the best approximation is the line $y = 9 + 4t$. □